$$\left.\frac{\partial f_1}{\partial t}\right|_{t=0} = i\zeta\left(\frac{\partial f_1(\zeta, 0)}{\partial \zeta} + z_1(\zeta, 0)\right)$$

With known $f_1(\zeta, 0)$ and $\partial f_1(\zeta, 0)/\partial t$ we can find $f_1(\zeta, t)$ from Eq. (2.1) and then integrating (1.14) with $b(t) = 0$, to determine $z_1(\zeta, t)$. In this case the unknown functions are sought in the form of Taylor's series. Formulas defining the solution of this problem are obtained by transition to the limit $r \to 0$ in (1.18).

It should be noted that all statements derived in Sect. 1 are valid in this limit case.

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BIBLIOGRAPHY

1. Fox, J. L. and Morgan, G. W., On the stability of some flows of ideal fluid with free surfaces. Quart. Appl. Math., Vol. 11, №4, 1954.
2. Courant, R., Equations with Partial Derivatives [Russian translation]. Mir, Moscow, 1964.

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ON ONE EXACT SOLUTION OF EQUATIONS OF PLAIN NONSTATIONARY MOTION OF GAS

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O. F. MEN'SHIKH
(Kuibyshev)
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An exact solution of equations of a plain nonstationary potential isentropic motion of gas dependent on two arbitrary functions with the Poisson adiabatic exponent equal to two is derived. The solution is interpreted as the motion of "shallow water" with a free surface which must be ruled. The general aspects of shallow water motion, and in particular the case of a cylindrical free surface an nonunivariate motion are considered.

1. The equation defining a plane nonstationary potential isentropic motion of gas is taken in the form [1]

$$(\Phi_H)^2 + \Phi_{uu}\Phi_{vv} - (\Phi_{uv})^2 - \Phi_H(\Phi_{uu} + \Phi_{vv}) + \qquad (1.1)$$
$$+ (\gamma - 1) H\, [(\Phi_{uH})^2 + (\Phi_{vH})^2 + 2\Phi_H\Phi_{HH} - \Phi_{HH}(\Phi_{uu} + \Phi_{vv})] = 0$$

where u and v are projections of the velocity vector $\mathbf{v}$ on the x and y axes of a Cartesian system of coordinates, H is the enthalpy, t is the time, γ is the Poisson adiabatic exponent, and Φ is the conjugate potential related to the velocity potential φ by formula

$$\Phi = \varphi - xu - yv + t\,[{}^1/_2\,(u^2 + v^2) + H] \qquad (1.2)$$

Transition to variables t, x, y is by formulas

$$t = \Phi_H, \qquad x = u\Phi_H - \Phi_u, \qquad y = v\Phi_H - \Phi_v \tag{1.3}$$

Equation (1.1) is derived on the assumption of functional independence of variables u, v and H. A nonzero Jacobian is required if the transition to the space of variables t, x, y is to be unambiguous.

For $\gamma = 2$ Eq. (1.1), after the substitution $\sqrt{H} = s$ becomes

$$\Phi_s [\Phi_{ss} - \Phi_{uu} - \Phi_{vv}] + s \{4 [\Phi_{uu}\Phi_{vv} - (\Phi_{uv})^2] + \\ + (\Phi_{us})^2 + (\Phi_{vs})^2 - (\Phi_{uu} + \Phi_{vv}) (\Phi_{ss})\} = 0 \tag{1.4}$$

Formulas (1.3) can now be rewritten as

$$t = {}^1/_2 s^{-1}\Phi_s, \quad x = {}^1/_2 s^{-1}\Phi_s - \Phi_u, \quad y = {}^1/_2 s^{-1}\Phi_s - \Phi_v \tag{1.5}$$

2. Let us seek particular solutions of Eq. (1.4) for which the over-all order of matrix

$$M = \begin{Vmatrix} \Phi_{ss} & \Phi_{su} & \Phi_{sv} \\ \Phi_{us} & \Phi_{uu} & \Phi_{uv} \\ \Phi_{vs} & \Phi_{vu} & \Phi_{vv} \end{Vmatrix}$$

is equal unity.

For the sake of definiteness we assume that relationships

$$\Phi_u = P_1 (\Phi_s), \qquad \Phi_v = P_2 (\Phi_s) \tag{2.1}$$

hold for the first derivatives of function Φ.

The particular solutions of (1.4) with condition (2.1) satisfied are solutions with differential relationships [2]. To separate all such solutions we use the following tangential transformation [3]:

$$T (\xi, u, v) = - \Phi + s\Phi_s, \qquad s = T_\xi, \qquad \Phi_u = - T_u \\ \Phi_v = - T_v, \qquad \xi = \Phi_s (s_\xi \neq 0) \tag{2.2}$$

The tangential transformation (2.2) applied to Eq. (1.4) and the subsequent representation of T in the form

$$T = \psi (\xi) - P_1 (\xi)u - P_2(\xi)v$$

yields all solutions of Eq. (1.4) with allowance for condition (2.1). Taking into account (1.2) and (1.5) we obtain the sought solution

$$u = t^{-1}[x + P_1 (\xi)], \quad v = t^{-1}[y + P_2 (\xi)] \tag{2.3}$$

$$\Gamma = P_1'x + P_2'y - \psi't + P_1P_1' + P_2P_2' + {}^1/_2\xi = 0 \tag{2.4}$$

$$\varphi = \xi\psi' (\xi) - \psi + {}^1/_2 t^{-1} [(x + P_1)^2 + (y + P_2)^2] - \\ - {}^1/_4 \xi^2 t^{-1} - \xi t^{-1}[(x + P_1)P_1' + (y + P_2)P_2'] \tag{2.5}$$

$$\xi = 2ts \tag{2.6}$$

Functions P_1 and P_2 are related by the Monge equation

$$(P_1')^2 + (P_2')^2 = 1 \tag{2.7}$$

whose solution is written in its final form as [4]

$$P_1 = \int \cos \theta (\xi) \, d\xi, \qquad P_2 = \int \sin \theta (\xi) \, d\xi \tag{2.8}$$

We have thus derived the particular solution of Eq. (1.4), which contains two arbitrary functions $\theta(\xi)$ and $\psi(\xi)$.

At fixed t Eq. (2.4) defines a particular case of a ruled surface generated by the motion of a straight line parallel to a given plane, i.e., a cylindroid [5]. This surface is developable (to wit, cylindrical) only for $\theta = \text{const}$.

3. Let us consider the derived class of solutions (2.3) - (2.6) as defining the motion of "shallow water" over a horizontal bottom. The enthalpy can then be represented in the form

$$H = gZ \tag{3.1}$$

where g is the acceleration of gravity and Z the height of the free surface of fluid.

Pressure p in the shallow water approximation varies hydrostatically along the vertical column of fluid [6]. The mean pressure $\langle p\rangle$ and "density" $\langle \rho\rangle$ of the fluid are related to the height of its free surface by formulas

$$\langle \rho\rangle = {}^1/_2 g\rho Z^2 = \int_0^Z p dz, \quad \langle \rho\rangle = \rho Z, \qquad H = 2\frac{p}{\rho} \tag{3.2}$$

It follows from the foregoind analysis that at any instant the liquid free surface is a ruled one, and that the arrangement of the straight lines over it is defined by functions $\theta(\xi)$ and $\psi(\xi)$. The ruled surface (2.4) is regular, since by virtue of that equation the partial derivatives with respect to x, y and ξ in the left-hand side of the latter cannot simultaneously vanish.

Let us find the conditions under which discontinuities of the shallow water free surface are possible. It follows from (2.6) and (3.1) that the variable ξ is defined by the formula $\xi = 2t\sqrt{gZ}$. Hence it is sufficient to determine the conditions of unboundedness of $\partial\xi/\partial x$ and $\partial\xi/\partial y$. From (2.4) follows that

$$\partial\xi/\partial x = -P_1'/\Gamma_\xi', \qquad \partial\xi/\partial y = -P_2'/\Gamma_\xi' \tag{3.3}$$

Owing to the boundedness of P_1 and P_2' it is necessary to equate the denominators in the last formulas to zero, which together with (2.4) yields (for fixed t) the system of two equations

$$\begin{gathered}\Gamma = (\xi)x + \beta(\xi)y + \omega(\xi) = 0 \\ \Gamma_\xi' = \alpha'(\xi)x + \beta'(\xi)y + \omega'(\xi) = 0 \\ \alpha(\xi) = \cos\theta, \quad \beta(\xi) = \sin\theta \\ \omega(\xi) = \cos\theta\int\cos\theta d\xi + \sin\theta\int\sin\theta d\xi - \psi' t + {}^1/_2\xi\end{gathered} \tag{3.4}$$

System (3.4) defines the envelope of the family of curves dependent on the single parameter ξ. Two cases are possible.

The first case

$$\begin{vmatrix}\alpha & \beta \\ \alpha' & \beta'\end{vmatrix} = \frac{d\theta}{d\xi} \neq 0 \tag{3.5}$$

In this case it is possible to find for fixed ξ and t from system (3.4) the coordinates of points pertaining to the envelope

$$\begin{gathered}x = -P_1 + r_1, \quad y = -P_2 + r_2 \\ r_1 = \frac{1}{\theta'}(\alpha_{11}t + \alpha_{12}), \qquad r_2 = \frac{1}{\theta'}(\alpha_{21}t + \alpha_{22}) \\ \alpha_{11} = \psi'(\xi)\theta'\cos\theta = \psi''(\xi)\sin\theta, \ \alpha_{12} = {}^3/_2\sin\theta - {}^1/_2\xi\theta'\cos\theta \\ \alpha_{21} = \psi'(\xi)\theta'\sin\theta + \psi''(\xi)\cos\theta, \ \alpha_{22} = -[{}^3/_2\cos\theta + {}^1/_2\xi\theta'\sin\theta]\end{gathered} \tag{3.6}$$

For given functions θ and ψ Eqs. (3.6) written in parametric form define the envelope of a family of straight lines. It is possible to construct this envelope a priori. Hence disruption of the shallow water free surface takes place along the straight line (3.6).

The second case occurs in the investigation of system (3.4), when the conditions

$$\theta' = 0, \qquad \psi'(\xi)t = {}^3/_2 \tag{3.7}$$

are simultaneously satisfied. The second of Eqs. (3.4) is then identically satisfied and the disruption of the shallow water free surface occurs along the generatrix of the ruled surface for certain values of ξ and t which are to be determined from system (3.7). Streamlines of the considered motion of shallow water (with t taken as parameter) are defined by the system of two equations

$$[x + P_1(\xi)]\frac{dy_1}{d\xi} - [y + P_2(\xi)]\frac{dx_1}{d\xi} = 0 \tag{3.8}$$
$$\alpha(\xi)x_1 + \beta(\xi)y_1 + \omega(\xi) = 0$$

Assuming that the second of Eqs. (3.4) is not valid in the considered region, from (3.8) we obtain the system

$$\frac{dx_1}{d\xi} = \frac{x_1 + p_1}{{}^1/_2\xi - \psi' t}\Gamma'_\xi, \qquad \frac{dy_1}{d\xi} = \frac{y_1 + P_2}{{}^1/_2\xi - \psi' t}\Gamma'_\xi \tag{3.9}$$

Equation (2.4) with (2.3) taken into account assumes the form

$$P_1'(\xi)u + P_2'(\xi)v + t^{-1}({}^1/_2\xi - \psi' t) = 0 \tag{3.10}$$

This shows that, when

$${}^1/_2\xi - \psi' t \neq 0 \tag{3.11}$$

then, by virtue of (2.7), u and v cannot simultaneously vanish. Thus, when (3.11) applies and $\Gamma_\xi' \neq 0$, system (3.9) has no singular points. It is possible to state for system (3.9) the Cauchy problem

$$\xi = \xi_0, \quad x_1 = x_1(\xi_0), \quad y_1 = y_1(\xi_0) \tag{3.12}$$

provided the input data do not lie on the straight line (3.6), which can always be assumed.

Let us turn our attention to one singularity of formulas (2.3) - (2.8) when $\xi = 0$. It can be established by a direct test that for $\xi = 0$ the conditions of streamline flow

$$\frac{\partial l}{\partial t} + u\frac{\partial l}{\partial x} + v\frac{\partial l}{\partial y} = 0$$

are satisfied along the straight line

$$l = \alpha(0)x + \beta(0)y + \omega(0) = 0 \tag{3.13}$$

This shows that motions of shallow water flowing along the straight line (3.13) over a dry bottom are possible. Formulas (3.13) implies that for $\psi'(\xi) \equiv 0$ the motion of shallow water occurs along stationary streamlines. It is therefore possible to state the problem of motion of shallow water in some curvilinear channel whose walls must be defined by the integration of system (3.9) for $\psi'(\xi) = 0$.

4. Let us consider the particular case of $\theta = \text{const}$.

After rotation of the coordinate system, the solution (2.3) - (2.8) can be written in the form

$$u = t^{-1}(x + \xi), \qquad v = t^{-1}y, \qquad x + {}^3/_2\xi - t\psi'(\xi) = 0 \tag{4.1}$$

$$\varphi = \xi\psi' - \psi + {}^1/_2 t^{-1}[(x + \xi)^2 + y^2] - {}^1/_4\xi^2 t^{-1} - \xi t^{-1}(x + \xi) \tag{4.2}$$

The free surface of liquid, as shown by (4.1), is in this case cylindrical.

The solution depends on a single arbitrary function. It should be noted that the above assumptions do not apply in the case of one-dimensional motion of gas ($v = 0$) when $\gamma = 2$, since the system of equations of gasdynamics does not admit the solution (4.1) (4.2) if the latter is dependent on an arbitrary function. The flow of liquid is symmetric about the axis $y = 0$.

Solution (4.1), (4.2) has a physical meaning in the neighborhood of the plane $y = 0$, since for $y \to \infty$ the absolute value of the velocity vector infinitely increases.

Let us pose the following problem. Let at $t = t_0$ the free surface of the liquid be specified as a function of x

$$t = t_0, \qquad \xi = \varepsilon_1(x) \qquad (t_0 > 0) \tag{4.3}$$

Let us assume for simplicity that ε_1 is a monotonic function which for $-\infty < x < \infty$ has first and second derivatives.

In this case function (4.3) can be reversed. Let the result of such reversion be as follows:

$$t = t_0, \qquad x = \Omega(\xi) \tag{4.4}$$

For $t = t_0$ the velocity field is uniquely defined with respect to a given free surface by formula (4.1). The arbitrary function $\psi'(\xi)$ is found from (4.1) and the last relationship. Substituting $\psi'(\xi)$ into (4.1), we obtain

$$x + {}^3/_2\,\xi = tt_0^{-1}\,[\Omega(\xi) + {}^3/_2\,\xi] \tag{4.5}$$

Let us find the condition under which disruption of the shallow water free surface is possible. We write the second of Eqs. (3.4) as

$${}^3/_2 = tt_0^{-1}\,[\Omega'(\xi) + {}^3/_2] \tag{4.6}$$

This implies the possibility of determining $t > t_0$, if condition

$$-{}^3/_2 < \Omega'(\xi) < 0 \tag{4.7}$$

is satisfied. From (4.6) we then find

$$tt_0^{-1} = [1 + {}^2/_3\,\Omega'(\xi)]^{-1} \tag{4.8}$$

The minimum value of the right-hand side of (4.8) must be taken (this determines ξ at the instant of discontinuity onset). Substituting (4.8) into (4.5), we find the value of x at which a discontinuity is initiated

$$x = [\Omega - \xi\Omega']\,[1 + {}^2/_3\,\Omega']^{-1} \tag{4.9}$$

If follows from (4.1) and (4.7) that formation of discontinuities is not possible, if the liquid free surface increases for $x > 0$ ($(x + \xi) > 0$) . In the opposite case discontinuities can occur. Condition (4.7) imposes limitations on the direction of the tangent to the free surface of liquid. These conclusions are in agreement with known results of investigations relative to the motion of shallow water [6].

The system of Eqs. (3.9) is integrable, and the equations of streamlines are as follows:

$$x_1 = -\,[{}^3/_2\xi + tt_0^{-1}\,[\Omega(\xi) + {}^3/_2\xi]\} \tag{4.10}$$

$$y_1 = cA(\xi)$$

$$A(\xi) = \exp\int \frac{{}^3/_2 - tt_0^{-1}(\Omega' + {}^3/_2)}{{}^1/_2\xi - tt_0^{-1}(\Omega + {}^3/_2\xi)}\,d\xi \qquad (c = \mathrm{const})$$

5. We note that from the point of view of group classification the solution (2.3) - (2.8) is partly invariant [7]. It is derived from a sub-group with invariants

$$J_1 = u - xt^{-1}, \quad J_2 = v - yt^{-1}, \quad J_3 = \sqrt{H}t$$

with the following relationships:

$$J_1 = J_1(J_3), \qquad J_2 = J_2(J_3)$$

Here H is the "redundant" function.

The particular case of $\theta = \text{const}$ in (2.3) - (2.8) corresponds to the simplest functionally-invariant solution of Eq. (1.4) of the form

$$\Phi = \Phi(h), \qquad h = \alpha_1 u + \alpha_2 v \pm \sqrt{\alpha_1^2 + \alpha_2^2}\, s \qquad (\alpha_1, \alpha_2 = \text{const})$$

which can be checked by direct calculation.

The group properties of the system of equations defining the motion of shallow water were investigated in [8], however the derived here solution (2.3) - (2.8) was not obtained in that paper.

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BIBLIOGRAPHY

1. Men'shikh, O.F., On nonstationary plane transonic flows of gas. Izv. Vyssh. Uchebn. Zavedenii, Matematika, №2,1970.
2. Ianenko, N.N., Theory of Compatibility and Methods of Integration of Systems of Nonlinear Equations in Partial Derivatives. Trudy 4 Vses. Matem. S"ezda, Vol.2, "Nauka", Leningrad, 1964.
3. Belen'kii, I.M., Introduction to Analytical Mechanics. "Vysshaia Shkola", Moscow, 1964.
4. Ianenko, N.N., Traveling waves in systems of quasi-linear equations. Dokl. Akad. Nauk SSSR, Vol.109, №1, 1956.
5. Pluzhnikov, I.S., Ruled Surfaces and Methods of their Analysis. Izd. Mosk. Instrument. Inst., 1964.
6. Stoker, J.J., Waves on Water [Russian Translation], Izd. Inostr. Lit., 1959.
7. Ovsiannikov, L.V., Group Properties on Differential Equations. Izd. SO Akad Nauk SSSR, 1962.
8. Gladyshev, M.T., Certain families of exact solutions of equations in the theory of shallow water, PMTF, №6, 1969.

Translated by J.J.D.